

Math Monstrosity, Packet 11

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1 General Instructions to Moderators

1.1 For everyone: question formatting specific to this tournament

Power is denoted by a black circle, ●. Buzzes before the circle should be awarded power. The question is not bolded before the powermark, so please make sure you're awarding power correctly.

If a question begins with "paper and pencil ready", it is a computation question. Please read such questions slowly and pause for 2-3 seconds between clues.

If, at any time during an equation, you see something like $\frac{\mathbf{THIS}}{2}$ or $\mathbf{THIS}(n)$, then the word **THIS** refers to the thing being asked for in the question. If you're comfortable enough with math that you know what's going on, please read that as "this function" or "this quantity" or whatnot. If you're not, you can either parrot pronouns used earlier in the tossup, or just say "this thing" or "this".

Pronunciation guides are *[in brackets and italics]*.

1.2 For people who don't know how to read math: how to read math

In general, spell acronyms out. I will make sure to include a reading guide if this is not the case.

Please read Greek letters as they are (for example, read ϕ as "phi" and not "the golden ratio", even if it represents the golden ratio), with the notable exception of \sum and \prod , as in $\sum_{n=1}^5$, which should be read as "the sum from $n = 1$ to 5 of".

Similarly, \int_a^b is "the integral from a to b " and $\lim_{n \rightarrow \infty}$ is "the limit as n approaches infinity".

In general, something of the form $f(x)$ or $\lambda(u, v)$ is a function, and should be read as " f of x " and "lambda of u and v " respectively, and not as " $f x$ " and "lambda $u v$ ".

Please read large and/or complex fractions by saying "in the numerator", reading the numerator, saying "in the denominator", reading the denominator, and then saying "end of fraction". For simpler fractions, like $\frac{a}{b^2+c}$, you can simply read " a over b squared plus c ".

Please read $\binom{a}{b}$ as " a choose b ", not as " a over b ".

If you are not familiar with a certain piece of mathematical notation, please do your best to describe it to the players; for example, if you don't know that A^T means "the transpose of A ", read it as " A to the power of T " or " A superscript T ". Most of the notation used in this tournament is common enough that such descriptions, using words like "subscript" and "superscript", should suffice. If there are any problems which use particularly arcane notation, I will make sure to provide a reading guide.

2 Tossups

1. The Ax-Kochen this theorem relates p -adic numbers to formal power series, and Dye's Theorem guarantees the existence of one of these for probability spaces. Testing whether one of these exists for any two graphs has not been proved to be either P or NP, and Skiena suggested that it is neither; one algorithm for finding one of these for two \bullet graphs is canonical labeling. The logarithm function notably provides one from the additive group of real numbers to the multiplicative group of positive real numbers, and an automorphism is one from a set to itself. For ten points, identify this term used to describe structure-preserving bijections.

Answer: **isomorphism** [prompt on homomorphism but do not accept or prompt on "homeomorphism"]

2. The expected value of a random variable has this property, but its variance does not. This property is held by a homothety centered at the origin. A function satisfying Cauchy's functional equation has this property if it is continuous at at least one point, if it is bounded on any interval, or if it is monotone on any interval. Functions must be \bullet additive and homogenous of degree 1 to satisfy this property. This property is held by differentiation and integration with a fixed starting point. This property fails to be held by polynomials whose degree exceeds 1. For ten points, identify this property held by a function like $f(x)=3x$.

Answer: **linearity** [accept word forms]

3. An alternate statement of this result is the statement that the infinite arithmetic progressions of integers form a Helly family. Range ambiguity resolution techniques are a special case of this result, and Hermite interpolation applies this result to univariate polynomials. This result was first demonstrated by \bullet Sunzi in the *Sunzi Suanjing*, and is named for his nationality. For ten points, identify this result from number theory stating that for a set of pairwise coprime numbers n_i , and a set of integers a_i , then the system of congruences $x \equiv a_i \pmod{n_i}$ [NOTE: \equiv is to be read as "is congruent to"] for all i has a unique solution.

Answer: **Chinese remainder theorem**

4. *Note to players: the answer is somewhat general.* One of these objects is half-filled with water in a painting often attributed to Jacopo de' Barbari which depicts the title figure drawing a diagram to demonstrate one of Euclid's theorems; that painting is the Portrait of Luca Paioli. One of these objects is depicted with a number of trapezoidal doors through which the heads and legs of several turtles protrude; that work is *Gravitation* by Escher. A large one of these objects is depicted on the left side of an engraving which also features a magic square; that is Dürer's \bullet *Melancholia*. For ten points, identify these objects, the Platonic kind of which are depicted in a diagram of the solar system by Johannes Kepler.

Answer: **polyhedra** [accept any specific polyhedra, prompt on something like "solids" or "geometric shapes"]

5. This country's first Antarctic expedition was named for one of its mathematicians. A mathematician from this country names a question mark function with fractal properties. Another mathematician from this country names a plane also known as the inversive plane. A mathematician from this country names the diagrams used to represent partially ordered sets, while another mathematician from this country names \bullet surfaces which are one-dimensional complex manifolds. One mathematician from this country names the cgs unit of flux density, while a physicist from this country names the SI unit of resistance. For ten points, identify this country home to Bernard Riemann and Carl Friedrich Gauss.

Answer: Germany [accept Deutschland, liberally accept historical names like "German Empire" or "Third Reich" and the same in German, prompt on Prussia and variants thereof]

6. The bidual version of this shape is a graph with 12 vertices, while the Bislit version has eight. When this shape is rotated about one of its space diagonals, the resulting envelope is a hyperboloid of one sheet. Prince Rupert's one of these shapes is the largest one of them which can be passed through a hole in another one of these shapes. This shape has Wythoff symbol $3|24$ and is the Wenninger model W_3 . This shape, which is an equilateral zonohedron, was shown by Turney to have 11 distinct nets. This shape cannot be stellated because all of its faces are mutually \bullet perpendicular. For ten points, identify this Platonic solid whose volume is equal to its edge length to the third power.

Answer: cube [accept regular hexahedron, prompt on hexahedron]

7. An investigation of a method named for this mathematician led David Robbins and Howard Rumsey to the alternating sign matrix conjecture. This man proposed a voting method using swaps to extend the Condorcet method, and was responsible for the first printed proof of the Rouché-Capelli theorem. This man formulated syllogisms to train students in symbolic logic. This man names a method of finding the determinant of a matrix in which one first finds the determinants of every 2-by-2 submatrix; this is his namesake \bullet condensation. For ten points, identify this mathematician and author who pseudonymously wrote *Through the Looking-Glass* and the poem *Jabberwocky*.

Answer: Charles Lutwidge Dodgson [prompt on Lewis Carroll]

8. These statements can be used to prove that there is only one group with 15 elements, and the third of these statements leads to an easy proof of Wilson's theorem. The second of these statements claims for any p that all maximal p -subgroups of a group are conjugate to each other. The first of these statements strengthens Cauchy's theorem, and claims that for every prime factor p with multiplicity n of the order of a group, there exists a \bullet maximal p -subgroup with order p^n . For ten points, identify these three fundamental results from finite group theory, named after a Norwegian mathematician who also names maximal p -subgroups.

Answer: Sylow's theorems

9. The Graham scan and SMAWK algorithms use this data structure to organize information, and the Burroughs B5000 was designed based around this data structure. Chef and FALSE are examples of programming languages based around this thing, which is also central to the Forth family of languages. Machines named for this structure are contrasted with ● register machines. The peek operation allows a program to see the top element of this structure without removing it. Possessing push and pop operations, for ten points, identify this data structure with last-in-first-out properties.

Answer: **stack**

10. A tacpoint is a tangent point of this many similar curves, and a stratified manifold is a manifold with certain properties in this many dimensions. Pólya developed the method of this many loci to construct geometric objects. The Maclaurin-Bézout theorem states that this many cubic curves intersect in ● nine points. The F-ratio is the ratio of this many different estimates of a normal distribution's variance. Every simple polygon has at least this many nonoverlapping ears, and graphs with this many points cannot contain any cycles. The field with this many elements has operations corresponding to logical XOR and logical AND. For ten points, give this number, the only even prime.

Answer: 2

11. *Pencil and paper ready.* $\int_0^{\text{THIS}} x^2 + 5 = \frac{200}{3}$. The product $\begin{bmatrix} 1 & 4 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ minus the identity matrix has this number in the lower right cell, and the derivative of $x^3 + x^2$ at this number is equal to 85. This number is the side length of a regular hexagon with area $\frac{75\sqrt{3}}{2}$. This number is equal to ● $\int_{-1}^3 |x|$, and it's the length of the semimajor axis of an ellipse with a focus at $(0, 3)$ and a vertex at $(4, 0)$. For ten points, identify this number, the middle value of the only prime triplet.

Answer: 5

12. Erret Bishop argued that this statement is constructively acceptable because its namesake concept is implied by the meaning of existence. This statement implies Krull's theorem and ensures that every partially ordered set has a maximal antichain. This statement implies that every vector space has a ● basis and that every product of compact spaces is compact. This statement is equivalent to the statement that every set has a well-ordering, as well as Zorn's lemma. For ten points, identify this controversial statement in set theory stating that the Cartesian product of nonempty sets is nonempty, which makes up the C in ZFC.

Answer: **axiom of choice** [prompt on anything equivalent to the Axiom of Choice like Zorn's lemma and well-ordering principle, unless it's already been read (because the first clue is unique to AC)]

13. The Bombieri one of these is defined as $\left[\sum_{i=0}^n \binom{n}{i}^{1-p} |a_i|^p \right]^{\frac{1}{p}}$ for a prime

number p . The Frobenius one of these is equal to $\sqrt{\text{Tr}(\mathbf{A}\mathbf{A}^H)}$. This quantity for quaternions is equal to the square root of a quaternion times its quaternion conjugate. The $\bullet \ell^1$ kind of these is equal to a sum of absolute values, and the most commonly used one of these, the ℓ^2 one, is equal to the square root of the sum of the squares of the components of a vector. For ten points, identify this type of quantity that generally is used to refer to the size or length of an object.

Answer: **norm** [prompt on absolute value or magnitude]

14. George Bergman constructed a p -adic analogue of this space. This space fails to satisfy the Lindelöf condition and does not possess a countable dense subset; moreover, despite the fact that this space is locally compact, it is not paracompact. This space, which is not second countable, is formed from the cartesian product of ω_1 with $[0, 1)$. Any open interval (a, b) of this space is homeomorphic to the interval $(0, 1)$ in the reals, but the entire space is not homeomorphic to any subset of any $\bullet \mathbb{R}^n$. For ten points, identify this one-dimensional manifold locally homeomorphic to the reals and formed from an uncountable number of half-open intervals.

Answer: **long line** [accept **Alexandroff line** or **long ray**]

15. One proof of this statement used the Durand-Kerner method and the homotopy continuation principle to achieve a constructive proof, and Rouché's theorem can be used to obtain a short proof. Almira and Romero showed that this statement's falsehood would require that the sphere be flat. The maximum modulus principle is central to a complex-analytic proof of this statement, which implies that every polynomial with complex coefficients can be \bullet factored down to monomials with complex coefficients. This theorem states that the field of complex numbers is algebraically closed. For ten points, identify this theorem stating that any complex-valued polynomial has at least one complex root.

Answer: **fundamental theorem of algebra** [prompt on fundamental theorem, do not accept or prompt on "fundamental theorem of [anything else]", including "fundamental theorem of linear algebra"]

16. A book by Kobayashi and Nomizu on differential geometry was described by J. Eells as "essentially a [one of these works]". One of these works contains descriptions of alligation, the rule of false position, and the rule of three; that work was written by Edward Cocker. The equals sign was introduced in another of these works by Robert Recorde entitled *Arithmetick: or, the Ground of Arts*. Robin Hartshorne's \bullet *Algebraic Geometry* is one of these works which dealt with scheme theory in a way accessible to graduate students. Sheldon Axler's *Linear Algebra Done Right* is an example of, for ten points, what type of work which also includes the Prentice Hall Mathematics series?

Answer: math **textbooks** [prompt on math books, accept answers indicating that they're used for **teaching** or **learning** or **instruction**]

17. The essential versions of these hold their desired properties everywhere but a set of measure zero. A lattice is a partially ordered set all of whose finite subsets

have these, and if the lattice is complete, then all of its subsets have these. If a linearly ordered set has the property named for this concept, and it is dense, it is called a linear continuum. A space is said to be Dedekind \bullet complete if every bounded subset of it contains one of these, and the set of rationals whose squares are less than two does not have one of these in the rationals. For ten points, identify this analytic concept which is the least upper bound of a set.

Answer: supremum [accept infimum until “least upper bound” because the clues are logically equivalent, accept least upper bound before read, accept greatest lower bound until “least upper bound” is read, do not accept or prompt on “maximum” or “minimum”]

18. An ordinal number α is referred to as a this ordinal if there does not exist an ordinal β such that $\beta + 1 = \alpha$. A Hopf bifurcation takes a fixed point to a cycle described by this term. Banach names one of these entities which is a linear functional f satisfying $\|f\| = f(1) = 1$ and $f(\{a_{n+1}\}) = f(\{a_n\})$. The direct and inverse types of this operation act on modules. This operation’s superior type performs this action on a \bullet sequence of upper bounds, and a set is closed iff it contains all of its points described by this term. A function f is continuous at a point x if $f(x)$ is equal to this operation of f as it approaches x . A sequence is convergent if, for ten points, what operation on its terms has a finite output value?

Answer: limit

19. This game, which is played with a fixed subset X of a topological space Y , is determined if X has the Baire property in Y . The second player has a winning strategy in this game if X is of the first category in Y , that is, if X is countable and Y is T_1 and has no isolated points. This was the first infinite-positional game of perfect information to be studied. Identify this topological game in which two players alternate in choosing progressively nested members of a family of subsets of a topological space in an attempt to have their infinite intersection cover, or not cover, a fixed subset of that space, for \bullet ten points.

Answer: Banach-Mazur game

20. Godsil and Royle showed that the fractional version of this quantity is equal to the fractional clique number, and that it is less than or equal to this quantity. Erdős proved that there exist graphs with arbitrarily large girth and this quantity. Brooks’s theorem states that for non-complete graphs that are not odd cycles, this quantity is at most equal to δ , the maximum vertex degree. This quantity is equal to n for a \bullet complete graph with n vertices. This quantity is usually denoted with the letter χ [*chi*], and for planar graphs this quantity is at most four. For ten points, identify this quantity from graph theory, which gives the minimum number of colors required to color a given graph.

Answer: chromatic number [prompt on descriptions like “the number of colors required to color a graph” or similar before the last clue is read]