Math Monstrosity, Packet 10

Conor Thompson
University of Michigan

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1 General Instructions to Moderators

1.1 For everyone: question formatting specific to this tournament

Power is denoted by a black circle, ●. Buzzes before the circle should be awarded power. The question is not bolded before the powermark, so please make sure you're awarding power correctly.
If a question begins with "paper and pencil ready", it is a computation question. Please read such questions slowly and pause for 2-3 seconds between clues.
If, at any time during an equation, you see something like $\frac{\textbf{THIS}}{2}$ or $\textbf{THIS}(n)$, then the word **THIS** refers to the thing being asked for in the question. If you're comfortable enough with math that you know what's going on, please read that as "this function" or "this quantity" or whatnot. If you're not, you can either parrot pronouns used earlier in the tossup, or just say "this thing" or "this".
Pronounciation guides are *[in brackets and italics]*.

1.2 For people who don't know how to read math: how to read math

In general, spell acronyms out. I will make sure to include a reading guide if this is not the case.
Please read Greek letters as they are (for example, read ϕ as "phi" and not "the golden ratio", even if it represents the golden ratio), with the notable exception of $\sum$ and $\prod$, as in $\sum_{n=1}^{5}$, which should be read as "the sum from $n = 1$ to 5 of".
Similarly, $\int_a^b$ is "the integral from a to b" and $\lim_{n\to\infty}$ is "the limit as n approaches infinity".
In general, something of the form $f(x)$ or $\lambda(u, v)$ is a function, and should be read as "$f$of x" and "lambda of u and v" respectively, and not as "f x" and "lambda u v".
Please read large and/or complex fractions by saying "in the numerator", reading the numerator, saying "in the denominator", reading the denominator, and then saying "end of fraction". For simpler fractions, like $\frac{a}{b^2+c}$, you can simply read "a over b squared plus c".
Please read $\binom{a}{b}$ as "a choose b", not as "a over b".
If you are not familiar with a certain piece of mathematical notation, please do your best to describe it to the players; for example, if you don't know that A^T means "the transpose of A", read it as "A to the power of T" or "A superscript T". Most of the notation used in this tournament is common enough that such descriptions, using words like "subscript" and "superscript", should suffice. If there are any problems which use particularly arcane notation, I will make sure to provide a reading guide.

2 Tossups

1. All rectifiable polyominoes are examples of these things. They're not fractals, but the Sierpinski triangle and Gosper island are both examples of these things. Karl Scherer and George Sicherman have discovered pentagonal examples of these things, in addition to the oldest-known pentagonal one, the sphinx. The term for these things was introduced by Simon Golomb as a pun on the name of a certain class of ● animals. All parallelograms and triangles are examples of these things with order 4, and all examples of these things lead to a tiling of the plane. For ten points, identify these shapes which can be tiled by identical smaller copies of themselves.

Answer: **rep-tiles**

2. Mordell names a special case of these where one value is an integer. Abelian varieties can be considered generalizations of these objects which cannot be parametrized by rational functions. Weierstrass names a function which describes how to obtain the algebraic expression of one of these if one has the geometric object on which all of its points lie. These objects are defined as the embeddings of tori on the complex projective plane. A conjecture stating that these objects are ● modular if they are rational was central to the proof of Fermat's Last Theorem. For ten points, identify these curves with equation $y^2 = a^3 + ax + b$.

Answer: **elliptic curves**

3. These entities are not subject to gimbal lock, making them more useful than Euler angles. These form the only non-trivial central simple algebra over the reals, and are one of the two classes of the Brauer group of the reals. These can be constructed by performing the Cayley-Dickinson construction on the ● complex numbers. Dividing one of these objects by its norm yields its versor. The Frobenius theorem guarantees that these form the largest finite-dimensional division ring, and they were the first discovered noncommutative division algebra. For ten points, identify these four-dimensional analogues to the complex numbers.

Answer: **quaternion**s

4. One of these entities is referred to as orthodox if its idempotents form a sub-one of these, and if every element of one of these objects is idempotent, it is referred to as a band. By the Krohn-Rhodes theorem, every finite one of these entities is a divisor of a wreath product of certain other algebraic objects. A set of functions from a set to itself, which is closed under composition, is an example of the transformation type of these objects. These entities are distinguished from groupoids by their possession of the ● associative property. For ten points, identify this algebraic object consisting of a set endowed with an associative binary operation.

Answer: **semigroup**s

5. Sárkőzy's theorem guarantees that for sufficiently large n, a certain function of n does not have this property. An integer n divides $a^n - a$ iff $p-1$ divides $n-1$ for prime divisors p of n and n has this property, by Korselt's criterion. A word has this property if none of its subwords consist of two duplicated subwords. The ● Möbius function of all numbers without this property is zero. A conjecture of Erdős states that for all $n > 4$, $2n$ choose n has this property. An expression is in simplest radical form if the number under the square root has this property. For ten points, identify this property held by numbers with no duplicated prime factors, that is, no factors which are perfect squares.

Answer: **squarefree**

6. In one tablet, the scribe Bêl-bân-aplu used three hooks to represent this concept, rather than the usual two wedges. This concept was first used in India in the Lokavibhāga, and the word śūnya was used to represent it. The word ● *zephyrum* was used for this concept in medieval Europe. It's not i, but Cardan solved cubic equations without using this concept. In the Mayan calendar, a tortoise shell was used to represent this concept. Brahmagupta was the first to give rules for computing with this concept, and Fibonacci introduced it to European mathematics in the *Liber Abaci*. For ten points, identify this concept, which represents the cardinality of the empty set.

Answer: **zero** [accept logical equivalents, prompt on "nothing" or "vacuum" or logical equivalents which don't make clear that it's the number zero, accept descriptive answers like "the fact that zero is a number" or "the idea that zero exists"]

7. A topological space on this entity is the initial object of the category of topological spaces with continuous maps, and this entity is the initial object of the category of sets and of the category of functions. The axiom of extensionality guarantees this entity's uniqueness in Zermelo set theory. In the space of extended real numbers, this entity is the only set whose infimum is ● greater than its supremum. Bourbaki introduced the most commonly used symbol for this entity, inspired by a letter in the Danish and Norwegian alphabets. Any statement about sets which begins "for any element of a set X" is true if the set X is this one. For ten points, identify this set which has cardinality zero because it has no elements.

Answer: **empty set** [accept **void set**, grudgingly accept **null set** even though that's really an unrelated measure theory term, accept descriptive answers like "the **set** with **no elements**" until the last clue is read]

8. This statement was first proved as a lemma on the way to proving the intermediate value theorem. This theorem can be easily derived from Cantor's intersection theorem, which deals with nested intervals. One way of stating this theorem is that every infinite bounded set in $\mathbb{R}^n$ has an ● accumulation point. A corollary of this theorem is the Heine-Borel theorem, that is, that any subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. For ten points, identify

this doubly-eponymous theorem which states that every bounded sequence has a convergent subsequence.

Answer: **Bolzano-Weierstrass** theorem [accept **sequential compactness** theorem as well as descriptions of its statement until "doubly-eponymous" and prompt afterward]

9. Jessen's [this] icosahedron is a shaky polyhedron in which faces of an icosahedron are replaced by isosceles triangles. Polynomials are described using this term if they obey the relationship that the integral from a to b of $w(x)p_m(x)p_n(x)$ is equal to c_n times the Kronecker delta of m and n. Circles in the plane are the group orbits of groups described by this term, of which the special kind is the component containing the identity. This condition for linear transformations can be expressed as $a_{ij}a_{ik}$ is equal to the Kronecker delta of j and k. Three confocal conic ● surfaces have this property, which can be attained by at most three collections of surfaces in three-dimensional space. For ten points, identify this property held by two members of a space if their inner product is zero.

Answer: **orthogonal**ity [accept word forms, prompt on **perpendicular**ity and word forms]

10. The Etruscan Venus surface and the Ida surface are three-dimensional shadows of an embedding of this structure in four-dimensional space. Frederick Winsor's *The Space Child's Mother Goose* includes a rhyme about "three jolly sailors" who went to sea in one of these structures, however "the scenery seen was exceedingly dull / as the sea was entirely ● inside the hull". This surface is the sole exception to the Heawood conjecture because its chromactic number is six, and it is nonorientable and has no boundary. This surface can be formed by adjoining two Möbius strips. For ten points, identify this surface named for a German mathematician which has no interior or exterior.

Answer: **Klein bottle**

11. A ring is called quasilocal if it is a unit ring with finitely many maximal ideals and also has this property. The snake lemma and nine lemma are lemmas pertaining to diagrams described by this term; such diagrams describe a collection of maps and consist of letters or other symbols joined by arrows. The Wolfram Language attribute for this property is ● "Orderless". This property is always held by the addition operation in a ring, but not by the multiplication operation. Inner products like the dot product have this property, but the cross product does not. For ten points, identify this property held by a binary operation $*$ if $a * b = b * a$ for all a and b.

Answer: **commutativity** [accept word forms]

12. With Tricomi, this mathematician names the partial differential equation $u_{xx} + xu_{yy} = 0$. In music theory, this man and Fokker name any musical scale whose pitches are generated from products of a set of prime numbers. There are either 65 or 66 of this man's idoneal numbers, depending on the truth of the Riemann Hypothesis. An identity named for this man states that the product

of two sums of four ● squares is itself a sum of four squares, and this man names, with Mascheroni, a constant equal to the limit of the difference between the harmonic series and the natural logarithm. For ten points, identify this Swiss mathematician, a formula named for whom is used to define complex exponentiation.

Answer: Leonhard **Euler**

13. *Description acceptable.* One method for performing this action can be extended using the Itô integral; that is the Adomian method. Another method for performing this action introduced the convergence-control parameter and is called the homotopy analysis method. The split-step method is used to do this, while the Frobenius method for doing this returns an infinite series. The method of ● characteristics is a method for performing this action. One method for doing this is called the method of undetermined coefficients, while a general method for doing this is variation of parameters. For ten points, identify this action in which a certain kind of equation is solved.

Answer: **solving differential equations** [prompt on partial, accept logically equivalent answers like "finding solutions to differential equations", accept answers pertaining to certain types of differential equations]

14. The Lucas correspondence relates the Hanoi graph to a graph derived from this object. One early reference to this object called it the Staircase of Mount Meru, and in some countries, this object is alternately named for Yang Hui or ● Khayyam. If part of this object is represented as a matrix, the determinant of that matrix will be 1. The diagonals of this object contain the figurate numbers of simplices with progressively higher dimension; this notably includes the triangular numbers. This object's entries correspond to coefficients of binomials. For ten points, identify this object in which every number is the sum of the two numbers above it.

Answer: **Pascal's triangle** [accept **Yang Hui's triangle** and **Khayyam's triangle** before their respective mentions]

15. The Arnoldi method is an application of this process to the Krylov sequence. This process allows the existence of a smooth deformation retraction of one type of Steifel manifold into the other. The HJLS algorithm is numerically unstable because of its reliance on this process, which is also numerically unstable as shown by Golub and Van Loan. This process can be used to diagonalize a nondegenerate symmetric bilinear form, and it can generate the Legendre ● polynomials over the open interval $(-1, 1)$ with weighting function 1. For ten points, identify this process used to generate an orthogonal basis from any linearly independent set.

Answer: **Gram-Schmidt** process [accept things like "algorithm" and such for "process"]

16. A paper in the Journal of Applied Mathematics by Lorenzi and Francaviglia discusses "Art and Mathematics" in this building. This building's columns,

which are capped with ellipsoids, are always twice as high in meters as the number of points in the cross-section of their base. A school built near this building for its construction workers' children features a wavy roof which is a ruled surface. This building uses hyperboloids for its windows in order to maximize the amount of light entering it, as this building's architect saw light as the symbol of ● God. A magic square with magic number 33, the age of Jesus at his death, lies on the Passion facade of this building. For ten points, identify this massive unfinished church in Barcelona, designed by Antoni Gaudí.

Answer: Basílica i Temple Expiatori de la **Sagrada Família** [accept Templo Expiatorio de la **Sagrada Familia**, accept Basilica and Expiatory Church of the **Holy Family**]

17. A highly-cited 1983 paper by Andersen describes a version of the shake algorithm named for this quantity, and a book by Ray Kinslow of Tennessee Tech talks about phenomena that occur at high values of this quantity. The Stokes type of this quantity is equal to $\frac{2a^2 g(\rho' - \rho)}{9\eta}$ *[NOTE: ρ and η are the Greek letters rho and eta].* The Fermi type of this quantity is equal to $\sqrt{\frac{2 * \text{Fermi energy}}{\text{mass}}}$. The Lorentz factor is given by the reciprocal of the square root of 1 minus ● this quantity's square over the square of the speed of light, and is dimensionless. An object for which gravity and drag are in equilibrium has attained the terminal form of, for ten points, what quantity equal to the derivative of position?

Answer: **velocity** [do not accept or prompt on speed]

18. The fourth cumulant is equal to the kurtosis times this quantity squared minus three times this quantity squared. The Lexis ratio is a ratio of two versions of this value, one of which is calculated from a set of Lexis trials, and another of which is calculated assuming Bernoulli trials. Bessel's correction is multiplied by this quantity to obtain the expected value of another version of this quantity. This quantity, which is equal to the second central ● moment, is defined t as the expected value of$(X - \mu)^2$. This quantity is the square of the standard deviation. For ten points, identify this quantity, which is a measure of the level to which a data set is dispersed from its mean.

Answer: **variance**

19. Martin Gardner developed a variant of this game to be played on a smaller board while preserving traditional rules, which he called its "mini" version. A book by Raymond Smullyan entitled *The [this] Mysteries of Sherlock Holmes* contains a number of retrograde analysis puzzles about this game. A strategy-stealing argument cannot be used for this game because of the existence of ● zugzwang *[tsoots-vang]* situations. The works *My System* and *[this game] Praxis* by Aron Nimzowitch are among the seminal works examining this game's middlegame phase, which is less studied than the opening or endgame. For ten points, identify this game played on an 8-by-8 board with pieces like bishops and pawns.

Answer: **chess**

20. This letter denotes a polynomial associated with graphs which is 1 for all complete graphs. This letter names a function the derivative of whose logarithm at certain points is the Weierstrass zeta function. A collection of subsets of a set X which includes X and is closed under countable union and complement is an ● algebra denoted by this letter. If a topological space is equal to the union of countably many subsets, then it is said to be this letter-compact. This letter is used to represent the standard deviation, and the capital version of it is used to denote summation. For ten points, identify this letter of the Greek alphabet, whose analogue in English is s.

Answer: **sigma**, σ